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Existence of a solution of the problem of optimal control of mines for minerals

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Abstract: Dealt with in the study are exhaustible resources - minerals. In the study will be created a mathematical model of a particular economic system - mine for minerals and will be proven the existence of solution of the problem of optimal control of mines. Also will be formulated necessary and sufficient conditions for optimal control of mines. In today's market environment, many of the parameters for optimum control problems are unspecified. To deal with this fuzziness in conditions and parameters will be used Intuitionistic fuzzy sets, defined from Atanassov in [5]. The algorithms for solution of intuitionistic fuzzy models of mine's optimal control are presented.

Keywords and phrases: intuitionistic fuzzy set, mine, minerals, optimal solution.

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1. Introduction

Nonrenewable resours is fundamentally limited in quantity and grade. Their exhaustion – now or later – poses some questions about effective spending.

Goals[8] in extracting the resource are similar to those regarding the extraction of other natural resources:

(a) maximize net present value;

(b) minimize the deviation between the amount of extracted resource and a contractually specified amount;

(c) maximize production and operational flexibility.

Operations research modeling applied to mining applications begin from 1960. Operations research models for strategic, tactical and operational levels of planning within the development and exploitation phases have been constructed and implemented. Briefly review some of the seminal works in these categories are given in [9].

In [24] are pointed models of optimal control of mines. Let us briefly outline them: Ramsey [25] analysed the consumption-saving decision; Hotelling [11,25] showed how one resource under depletion can be controlled optimally; Allen included a chapter on the calculus of variations in his textbook on mathematical economics [3]; Stavins [26] used a dynamic analysis of maintenance of natural resources. Lozada applied to the Hotelling's model a new fundamental equation for the time intensity of the function change of optimal values of optimal control problems [14]. Lyon [15] discussed the role of ''costate variable'' (''shadow price'') for exhaustible and non-exhaustible resources. Piazza and Rappaport, in [23] considered the optimal extraction problem of resources that cannot be maintained continuously.

A methodological basis of optimal control of the economy is the mathematical theory of optimal processes, developed by a team of mathematicians under the guidance of Pontryagin [36]. To formulate specific problem of optimal control is necessary to define the set of variables describing the state of the economic system, to set the objective function and the system of restrictive conditions.

The study is organized as follows: in section 2 will be considered a general problem of the optimal control. In section 3 will be created a model of an optimal control of a mine for mineralls. In section 4 will be formulated necessary and sufficient conditions for optimal control of a mine. In section 5 will be proven existence of a solution of the presented problem. In section 6, the model will be extended by adding of profit's taxatio. In section 7 will be pointed ideas for intuitionistic fuzzy models.

2. General problem of optimal control

2.1. Formulation of the general problem

Let us consider the following problem for minimum [36]:

$$min \quad \int_{t_0}^{t_1} f^0(t,x,u)dt + \Phi(t_1, x(t_1)) \tag{1}$$

with restrictions:

$$u \in \Omega \subset R^m, \dot{x} = f(t,x,u), x(t_0) = x_0 \ (x_0 \in X_0), x(t_1) \in X_1,$$

where

$$u = u(t), u \in R^m, x = x(t), x \in R^n, t \geq t_0,$$
$$\{f(t,x,u), f^0(t,x,u)\}: R \times R^n \times R^m \to R^n;$$

$\Phi(t,x): R \times R^n \to R, X_0, X_1$ are mathematical diversities.

For the convenience let $\dot{x}^{n+1} = f^0(t,x,u)$ and $x^{n+1}(t_0) = 0$. Then the problem (1) can be described as follows in an equivalent way if $\Phi(t_1, x(t_1)) \equiv 0$:

$$\min x^{n+1}(t_1)$$
$$\dot{\tilde{x}} = \tilde{f}(t,x,u)$$
$$u \in \Omega, \tilde{x}(t_0) \in \widetilde{X_0}, \tilde{x}(t_1) \in \tilde{X}_1$$

2.2. Input data of the problem of the optimal control:

(1) a description of the object of control;

(2) an initial state of a physical system and the purpose of management;

(3) a class of admissible controls;

(4) a criterion of quality – a functional, which gives us a quantitative evaluation of the effectiveness of management.

The object of control is described as follows:

$\dot{x}_i = f^i(t, x^1, \dots, x^n, u^1, \dots, u^m), \quad i = 1, \dots, n$ (2)
$x \equiv x(t) = (x^1(t), \dots, x^n(t))$ – characterize the state of the object;
$u \equiv u(t) = (u^1(t), \dots, u^m(t))$ – characterize its control;
$X_0, X_1 \in R^n$ are initial and target sets;
$x_0 \in X_0$ is initial point.

The goal of management is to align the object with an initial $x_0 \in X_0$ in a point $x_1 \in X_1$. We will assume that the target set X_1 continuously depend on t and is a compact.

The class of admissible controls consists of measurable functions $u(t), for\, t_0 \leq t \leq t_1$, satisfying the restriction $u(t) \subset \Omega \subset R^m$, all of which brought the object from the starting point x_0 in one of the points of the target set X_1.

Quality criterion:

$$C(u) = \int_{t_0}^{t_1} f^0(t, x(t), u)t))dt,$$

where $\left\{f^0(t,x,u), \frac{\partial f^0}{\partial x}(t,x,u)\right\}$ are continuous in R^{n+m+1} functions.

Definition (1): We say that the problem of an optimal control is autonomous if:

a) the object of management is described by the system of ordinary differential equations of the form:

$$\dot{x}_i = f^i(t, x^1, \dots, x^n, u^1, \dots, u^m), \quad i = 1, \dots, I,$$

where $f(x,u) \in C^1((R^n \times \Omega)$;

b) Quality criterion has the form:

$$C(u) = \int_{t_0}^{t_1} f^0(x(t), u(t))dt,$$

where $f^0(x,u)) \in C^1(R^n \times \Omega)$ and $x(t)$ is the decision of (2).

Definition (2): The control $u^*(t)$ of the class Δ will call optimal in terms of the criterion of quality $C(u)$, if $C(u^*) \leq C(u)$ for each $u(t) \in \Delta$.

2.3. Principle of maximum of Pontryagin and conditions of transverzalnost

Let $u(t)$ is a control of Δ and the corresponding decision is

$$x(t) = \left(x^i(t)\right), \qquad i = 1, \dots, n.$$

Consider the vector $\tilde{x}(t) = \left(x^0(t), x(t), x^{n+1}(t)\right)$, which is the decision of the extended system:

$$\begin{gathered} \dot{\tilde{x}}^i = \tilde{f}(\tilde{x}, u), \\ \tilde{x}(t_0) = (0, x_0, t_0) \text{ i.e.} \\ \dot{x}^0 = f^0(x, x^{n+1}, u), \\ \dot{x} = f(x, x^{n+1}, u), \\ x^{n+1} = 1,\ \tilde{x}(t_0) = (0, x_0, t_0). \end{gathered} \tag{3}$$

Let to introduce the following concepts: a conjugated system and its decision, a function of Hamilton, a smooth diversity and a tangent space to a smooth diversity.

The *extended conjugated system* of (3) has the form:

$$\dot{\tilde{\eta}} = -\tilde{\eta} \frac{\partial \tilde{f}}{\partial \tilde{x}}\left(\tilde{x}(t), u(t)\right) \tag{4}$$

or

$$\dot{\eta}_0 = 0$$

$$\dot{\eta}_j = -\sum_{i=0}^{n} \eta_i \frac{\partial f^i}{\partial x^j}(x(t), t, u(t)), \qquad j = 1, \dots, n$$

$$\dot{\eta}_{n+1} = -\sum_{i=0}^{n} \eta_i \frac{\partial f^i}{\partial t}(x(t), t, u(t)).$$

$(n+2)$ −dimensional vector

$$\tilde{\eta}(t) = \left(\eta_0(t), \eta_1(t), \dots, \eta_{n+1}(t)\right), t_0 \leq t \leq t_1$$

is called *conjugate solution* of $\tilde{x}(t)$, if $\tilde{\eta}(t)$ is a non-trivial solution of the extended conjugated system (4).

Hamilton function for extended system is:

$$\widetilde{H}(\tilde{\eta},\tilde{x},u) = \eta_0 f^0(x,x^{n+1},u) + \cdots + \eta_n f^n(x,x^{n+1},u) + \eta_{n+1}.$$

Let denote with: $\widetilde{M}(\tilde{\eta},\tilde{x}) = \max_{u\in\Omega} \widetilde{H}(\tilde{\eta},\tilde{x},u)$;

$$\tilde{x} = (\hat{x},x^{n+1}),\ \text{where}\ \hat{x} = \hat{x}(t) = \left(x^0(t),x^1(t),\dots,x^n(t)\right)$$

$$\tilde{\eta} = (\hat{\eta},\eta_{n+1}),\ \text{where}\ \hat{\eta} = \hat{\eta}(t) = \left(\eta_0(t),\eta_1(t),\dots,\eta_n(t)\right)$$

$$\widetilde{H}(\tilde{\eta},\tilde{x},u) = \widehat{H}(\hat{\eta},\hat{x},t,u) + \eta_{n+1};\ \widetilde{M}(\tilde{\eta},\tilde{x}) = \widehat{M}(\hat{\eta},\hat{x},t) + \eta_{n+1}$$

$$x^i \equiv x^i(t), i = 0,\dots.,n+1$$

$$\eta_i \equiv \eta_i(t), i = 0,\dots.,n+1$$

Definition (3): Under smooth diversity will understand set assigned finite number of equations of the kind:

$$g_i(x) = 0, i = 1,\dots,k,\ \ x \in R^n, k < n,$$

where the functions $g_i(x)$, $i = 1,\dots,k$ are continuously differentiable.

Definition (4): Let the smooth diversity S has the form:

$$g_i(x) = 0, i = 1,\dots,k,\ \ x \in R^n, k < n,$$

where $g_i(x), i = 1,\dots,k$ are continuously differentiable. Under a tangent space to the smooth diversity S in a point x_0 of S will call a set, satisfying the following system of equations:

$$\langle \nabla g_i(x_0), x - x_0\rangle = 0, i = 1,\dots,k.$$

Note: In this study we will use only smooth diversities.

Definition (5): Present value of a payment [35]

We denote the payment at a time t by the function $m(t)$.

The present value of the payment $m(t)$ is $m(0)$: $m(0) = m(t)e^{-rt}$.

Theorem 1:(Principle of maximum Pontryagin) [36]

Let us consider a controlled process in R^n:

$$\dot{x} = f(x,t,u) \tag{5}$$

Let Δ is the aggregation of all limited, measurable controls $u(t) \subset \Omega \subset R^m$, set of different terminal intervals $t_0 \le t \le t_1$ and transfer a starting point of the primary set X_0 at an end point of the target set X_1.

Let

$$C(u) = \int_{t_0}^{t_1} f^0\left(x(t),t,u(t)\right)dt$$

is a functional quality on the set of controls $u(t)$ of Δ for interval $t_0 \le t \le t_1$ with corresponding trajectories $x(t)$. If the control $u^*(t)(t_0^* \le$

$t \leq t_1^*$) with a corresponding decision $\tilde{x}^*(t)$ of the expanded system is optimal control in Δ, it exists nontrivial solution $\tilde{\eta}^*(t)$ of the extended conjugated system (4), such that

$$\widetilde{H}(\tilde{\eta}^*(t), \tilde{x}^*(t), u^*(t)) = \widetilde{M}(\tilde{\eta}^*(t), \tilde{x}^*(t))$$

almost everywhere and

$$\widetilde{M}(\tilde{\eta}^*(t), \tilde{x}^*(t)) \equiv 0, \eta_0^* \leq 0 \text{ for all } t: t_0^* \leq t \leq t_1^*.$$

This can also be written in the form:
$\widehat{H}(\hat{\eta}^*(t), \hat{x}(t), u^*(t)) = \widehat{M}(\hat{\eta}^*(t), \hat{x}^*(t))$ almost everywhere and

$$\widehat{M}(\hat{\eta}^*(t), \hat{x}^*(t), t) = \int_{t_0^*}^{t} \sum_{i=0}^{n} \eta_i^*(s) \frac{\partial f^i}{\partial t} (x^*(s), s, u^*(s))\, ds.$$

Furthermore $\eta_{n+1}^*(t_0^*) = \eta_{n+1}^*(t_1^*) = 0$ and therefore

$$\widehat{M}(\hat{\eta}^*(t_1^*), \hat{x}^*(t_1^*), t_1^*) \equiv 0.$$

If X_0 and X_1 are diversities in R^n with a tangent space T_0 and T_1 in the points x_0^* и x_1^* accordingly, the decision $\widetilde{\eta^*}(t)$ needs to be selected so as to satisfy the following conditions for transverzalnost (or only one of them): $\eta^*(t_0^*) \perp T_0$ or $\eta^*(t_1^*) \perp T_1$.

3. An economic staging of the problem of an optimal control of mines

As mention in [21], manager of the mine must decide whether to leave the resource in the ground or to extract and sell it at some price (P). If he conserves it, he can sell it in the future. If he exploits it, he can invest the proceeds at a positive rate of interest. In continuous time, he will be indifferent to saving or seling if:

$$\frac{\dot{P}}{P} = r \tag{6}$$

where $\dot{P} = \frac{\partial P}{\partial t}$ and r is the continuous compounding rate of interest. The condition of indifference is an equilibrium condition for suppliers to the market for the extracted product. It is known as Hotelling's rule for pricing a resource that is strictly limited in the supply. Hotelling' s rule [11] is a portfolio balance condition for all assets that can be freely shifted between portfolios in competitive markets and are sterile in the sense that the holding of the stock of the asset per se yields no net benefits. The price (P) rises at the rate of interest as the resource is extracted:

$$P = P(0)e^{rt}.$$

We suppose that net benefits accrue as a continuous flow and then

$$V = \int_{t=0}^{T} e^{-rt}\pi_t,$$

where V is the present value of the resource, π_t is the net benefit in the t−th period, r is the discount rate for continuous compounding and e^{-rt} is the continuous discount factor applied to the t−th period's net benefit [27,34].

Let us introduce the following indications:

$b \equiv b(t)$ – resource stock; $a \equiv a(t)$ – used force to extract the resource; $x \equiv x(t)$ – derived product; $p \equiv p(t)$ – market price; $q \equiv q(t)$ – resource price; $w \equiv w(t)$ – market price of the efforts.

3.1. Production function

In general terms each production function can be represented in the following form:

$$f\left(X_1, \dots X_i, \dots, X_m, Y_1, \dots Y_j, \dots, Y_n\right) = 0,$$

where X_i $(1 \leq i \leq m)$ is the value of a resource of the i-th appearance in the creation of a set of production results Y_j $(1 \leq j \leq n)$; Y_j is the magnitude of the production resulting from the j-th kind $(j = 1, \dots, n)$; m is the number of the used types of resources; n is the number of the species production results; f – the form of a multiple functional relationship between these resources and results.

The function of producing of mines is:

$$x(t) = F\big(a(t), b(t)\big),$$

where $a(t)$ is the used force to extract the resource and $b(t)$ is the resource stock. The units of the mining effort are marginally more productive when there is access to larger stocks.

Then the resource constraint is:

$$b = s - \int_0^t x dt,$$

where $s = b_0$ is the original stock of resource. So

$$x(t) = F\left(a(t), s - \int_0^t x dt\right).$$

Use $s - b(T) = \int_0^T x dt$ to compute T (the last moment of the existing the mine).

We assume that F:

- F is strictly increasing and separately concave in its arguments: F_a = marginal physical product of $a = MPP_a > 0$;
- F_b = marginal physical product of $b = MPP_b > 0, F_{aa} < 0, F_{bb} < 0$, and $F(0, b) = 0$ for each b [3].

Note: $F_a > 0$ as the growth of the factor a leads to an increase in manufacturing output.

b) MRS_b is a diminishing marginal rate of substitution along an isoquant (combinations of a and b so that x is constant).

$MRS_{ba} = -\frac{da}{db} = -$slope of an isoquant;

Let $\frac{d}{db} MRS_{ba} < 0$. From the implicit function theorem [31] we get, that $MRS_{ba} = \frac{F_b}{F_a}$.

c) $F_{ab} > 0, F_{ba} > 0$ (an increase in one input will increase the marginal product of the other).

$F_{ab} = F_{ba} > 0$ as mining effort is more productive when there is access to larger stocks and resource stocks are more productive when operated by a more intense effort.

3.2. Function of the expense

Function of the expense $C: R^2 \to R^+ \cup \{0\}$ will be defined as $C = wa$, where $w = w(t)$ is the market price of the effort a. It is used to provide operational cost of the spent a то exploited b.

As $x = F(a, b)$ и $F_a \neq 0$, then the implicit function theorem [4] allows us to express a as $a = a(x, b)$. From the same theorem implies that there is $a_x, a_b, a_{bb}, a_{xx}, a_{xb}, a_{bx}$ and their continuity.

So $C = wa = C(x, b)$. It follows that

$$C_x = \text{marginal cost} = wa_x = \frac{w}{F_a} > 0, C_{xx} = wa_{xx} > 0,$$

$$C_b = wa_b = -w\frac{F_b}{F_a} < 0, C_{xb} = wa_{xb} < 0$$

(marginal cost is increase).

The marginal cost is reduced if there is more b (the marginal cost of the mining a tonne of the ore is lower if there is more of it available to be mined) – $C_{bb} = wa_{bb} > 0$. The addition of b will reduce cost more if the marginal physical product of b is large, relative to that of a.

From $x = F(a, b), a = a(x, b)$ and the implicit function theorem follows that

$$da = a_x dx + a_b db$$

$$\text{or } dx = \frac{1}{a_x} da - \frac{a_b}{a_x} db;\ dx = F_a da + F_b db.$$

From these, $a_x = \frac{1}{F_a} > 0$, $a_b = -\frac{F_b}{F_a} < 0$.

From $a_x = \frac{1}{F_a}$ we find, that $a_{xx} = -\frac{F_{aa}}{F_a^3} > 0$ and

$$a_{xb} = -\frac{F_{aa}a_b + F_{ab}}{F_a^3} = -\frac{F_{ab} - F_{aa}F_b}{F_a^3} < 0$$

($F_{ab} > 0$ from 3.1c).

From $a_b = -\frac{F_b}{F_a}$ follows that

$$a_{bb} = -\frac{F_a F_{ba} a_b + F_a F_{bb} - F_b F_{aa} a_b - F_b F_{ab}}{F_a^3} = -\frac{F_{bb} F_a^2 - 2F_a F_b F_{ab} + F_{aa} F_b^2}{F_a^3} > 0.$$

Therefore $C_{xx} = wa_{xx} > 0, C_{xb} = wa_{xb} < 0, C_{bb} = wa_{bb} > 0.$

3.3. Math profit of the mine

Let with π_t we denote the profits of the mine at a time moment t. The profit of the mine at a moment t is the difference between the total revenue and the total expenditure at the same moment t.The total revenues from the sales of the extracted product at the moment t are $px = pF(a,b)$. The price of x depends negatively on the amount being sold. So $p = p(x)$, $p'(x) < 0$. The total cost of the mine at the moment t are $wa = C(x,b)$.

Then $\pi_t = pF(a,b) - wa = px - C(x,b)$. We will assume that the mine begin operating at the moment $t = 0$, so-called "current time". With T will mark the moment of closing the mine due to the inefficiency of the production. The profit of the mine at a time moment t, relative to current (zero) point is $\pi_0 = \pi_t e^{-rt}$. The aim of our management will be as follows the discounted present value of the mine for the time interval [0, T] be maximum i.e.

$$\max \int_0^T (pF(a,b) - wa)\, e^{-rt} dt$$

or corresponding dual expression

$$\max \int_0^T \left(px - C(x,b)\right) e^{-rt} dt$$

3.4. Resource pricing

b is stock and the quantity available to the industry is absolutely fixed at any given time. The price is determined from moment to moment [20] by the "forces of the marketplace." These forces balance the demand of all the firms with the given supply through the adjustment of the price (q).

3.5. Transversality conditions

The transversality condition "asset test" requires that a mine should be abandoned when it has no market value. The value of the mine, when abandoned is

$$qb = 0 \text{ at } T \text{ (terminal time)}$$

i.e. when the ore has no value or the ore is exhausted. If it pays to extract the first unit of the ore, it also pays to extract the last unit of ore. This condition will be satisfied if $b = 0, q > 0$ at T.

The other transversality condition "performance test" requires that net cash flow minus the value of depletion is

$$\pi - q\dot{b} = 0 \text{ at } T,$$

where $\pi = TR - TC = pF(a,b) - wa$ in the primal or $\pi = px - C(x)$ in the dual.

$\pi - q\dot{b}$ is interpreted in [21] as a "snapshot" performance indicator to be maximized at every point in time. If the maximum value of performance is zero, it is time to close the mine.

3.6. Resource stock

The resourse b satisfies the dynamic constraint: $\dot{b} = G(b) - F(a,b) = G(b) - x$, $G(b)$ is natural growth. The rate of extraction is $x = F(a,b)$. An operating mine always decumulates b since $G(b) = 0$.

3.7. Formulation of the problem

The production problem of the mine [21] is to maximize its own present value, which is the total resulting from adding up the discounted reserves from the sales of extractions from known quantity of the homogeneous ore. The mine is on an equilibrium extraction path when its owners are willing to hold its existing stock of ore in the ground.

In formulating the problem of an optimal control of a mine, we can approach it in two ways, depending on by what we want to manage - by the size of the effort that must be invested in the mine to extract minerals or by the amount of product we want to sell.

In the first case:

3.7.1. The object of management is the stock of minerals. It satisfies the ordinary differential equation

$$\dot{b} = -F(a,b), \quad F: R^2 \to R^+ \cup \{0\},\ a \equiv a(t), b \equiv b(t);$$

Initial point is $b(0) = b_0$.Therefore $b(t) \in [0, b_0]$ for each $t \in [0, t_2]$; The target set is $X_1 = R^+ \cup \{0\}$;

The class $\triangle$ of the admissible controls consists of the measurable functions $a(t): a(t) \geq 0, 0 \leq t \leq t_2$;

Criterion of quality:

$$V(a) = -\int_0^T (pF(a,b) - wa)e^{-rt} dt.$$

Then the problem became the following form:

$$min - \int_0^T (pF(a,b) - wa)e^{-rt}dt$$

$$\dot{b} = -F(a,b);\ \ b(0) = b_0; b(t) \in [0, b_0];\ a(t) \ge 0\, for\ 0 \le t \le t_2.$$

In the second case:

3.7.2. The object of the management be a set by the equation $\dot{b} = -x; x: R^2 \to R^+ \cup \{0\}$; x is industry sales. The remaining stock is $\dot{b}$.

It is reduced at a rate equal to sales and $\dot{b} < 0$. Initial point is $b(0) = b_0$.

Therefore $b(t) \in [0, b_0]$ for each $t \in [0, t_2]$; the target set is $X_1 = R^+ \cup \{0\}$; the class $\triangle$ of admissible controls consists of measurable $x(t): x(t) \ge 0, 0 \le t \le t_2$;

Criterion of quality:

$$V(x) = -\int_0^T (px - C(x,b))e^{-rt}dt.$$

Then the problem became the following form:

$$\min - \int_0^T (px - C(x,b))e^{-rt}dt$$

$$\dot{b} = -x;\ b(0) = b_0;\ b(t) \in [0, b_0];\ x(t) \ge 0\, for\ 0 \le t \le t_2.$$

4. Necessary and sufficient conditions for optimal management of mines

4.1. Applying the principle of maximum of the Pontryagen to the problem for an optimal control of a mine

Let us consider the autonomous process:

$$\dot{b} = -F(a,b) \text{ respectively } \dot{b} = -x$$

with measurable controls $a(t)$ respectively $x(t)$. On the set of controls $a(t)$ of $\triangle$ ($x(t)$ of $\triangle$) from the interval $0 \le t \le t_2$ with the respective trajectories $b(t)$ we will determine functional of quality

$$V(a) = -\int_0^T (pF(a,b) - wa)e^{-rt}dt$$

or

$$V(x) = -\int_0^T (px - C(x,b))e^{-rt}dt.$$

From the principle of maximum of Pontryagin we obtain that [31,32,35], that if $a^*(t)(x^*(t))$ is optimal control $(0 \le t \le T)$ with corresponding decision $\hat{b}^*(t) = (b^{0*}(t), b^*(t))$:

$$\dot{b}^0 = \frac{\partial \hat{H}}{\partial \eta_0} = -(pF(a,b) - wa)e^{-rt}, \ \ b^0(0) = 0, \dot{b} = -F(a,b), \ \ b(0) = b_0$$

respectively

$$\dot{b}^0 = -\big(px - C(x,b)\big)e^{-rt}, \ \ b_0(0) = 0, \dot{b} = -x, \ \ b(0) = b_0,$$

it exists nontrivial solution of the extended conjugated system

$$\dot{\eta}_0 = 0, \dot{\eta} = -\frac{\partial H}{\partial b} = \eta_0 \frac{\partial}{\partial b}[(pF(a,b) - \omega a)e^{-rt}] + \eta \frac{\partial F}{\partial b}(a,b),$$

respectively

$$\dot{\eta}_0 = 0, \dot{\eta} = -\frac{\partial H}{\partial b} = \eta_0 \frac{\partial}{\partial b}\big[\big(px - C(x,b)\big)e^{-rt}\big] - \eta \frac{\partial x}{\partial b},$$

such that:

1) $\hat{H}\big(\hat{\eta}^*(t), \hat{b}^*(t), a^*(t)\big) = \hat{M}\left(\hat{\eta}^*(t), \hat{b}^*(t)\right)$ almost everywhere
2) $\hat{M}\big(\hat{\eta}^*(T), \hat{b}^*(T), T\big) = 0$
3) $\eta_0^* \le 0$ everywhere for $0 \le t \le T$.

The target set X_2 for our problem is $[0, b_0]$, therefore the conditions of transverzalnost transformed into

4) $\eta(T) = 0$.

$$\hat{H}\big(\hat{\eta}, \hat{b}, a\big) = \eta_0[-(pF(a,b) - wa)e^{-rt}] + \eta(-F(a,b)),$$

respectively

$$\hat{H}\big(\hat{\eta}, \hat{b}, a\big) = \eta_0[-(px - C(x,b))e^{-rt}] + \eta(-x)).$$

We can assume that $\eta_0 \equiv \eta_0(t) = -1$. Then $\hat{H}\big(\hat{\eta}, \hat{b}, a\big) = \pi_t e^{-rt} - \eta F(a,b)$. For the time T, from 1), 2) and 4) we obtain that $\pi_T e^{-rt} = 0$. Then $\dot{\eta} = -pF_b e^{-rt} + \eta F_b$.

Let $q = \eta e^{rt}, q \equiv q(t)$ (η is the present value of q).

We find that $\dot{q} = rq - F_b(p - q)$ respectively $\dot{q} = rq + C_b$.

Moreover, the principle of maximum Pontryagin requires:

$$\hat{H}\big(\hat{\eta}^*, \hat{b}^*, a\big)\big|_{t=T} = 0 \text{ i.e. } [pF(a^*, b^*) - wa^* - qF(a^*, b^*)]|_{t=T} = 0$$

respectively

$$[px^* - qx^* - C(x^*, b^*)]|_{t=T} = 0.$$

Therefore if $a^*(t)\big(x^*(t)\big)$ is an optimal control for $0 \le t \le T$ with a corresponding $\hat{b}^*(t)$, it exists a nontrivial solution q of

$$\dot{q} = rq - F_b(p - q), \ \ (\dot{q} = rq + C_b), \tag{7}$$
$$q(T) = 0, \quad (q(T) = 0), \tag{8}$$

such that
$$[pF(a^*, b^*) - wa^* - qF(a^*, b^*)]|_{t=T} = 0,$$

$$\max_a [pF(a, b) - wa - qF(a, b)]\, e^{-rt} = \big(pF(a^*, b) - wa^* - qF(a^*, b)\big)e^{-rt}$$

Respectively
$$\begin{cases} [px^* - qx^* - C(x^*, b^*)]|_{t=T} = 0, \\ \max_x [px - C(x, b) - qx]\, e^{-rt} = \big((p - q)x^* - C(x^*, b)\big)e^{-rt}. \end{cases}$$

The solution of the problem of Cauchy [31,32] for q is:
$$q(t) = e^{\int_0^t (r+F_b)d\lambda}\left(q_0 - \int_0^t \left(F_b p e^{-\int_0^\lambda (r+F_b)ds}\right) d\lambda\right), \ t \in (0, T) \tag{9}$$
$$q(T) = 0 \tag{10}$$
$$q(0) = \int_0^T \left(F_b p e^{-\int_0^\lambda (r+F_b)ds}\right) d\lambda \tag{11}$$

respectively
$$q(t) = e^{\int_0^t r d\lambda}\left(q_0 + \int_0^t \left(C_b e^{-\int_0^\lambda r ds}\right) d\lambda\right), \ t \in (0, T) \tag{9*}$$
$$q(T) = 0 \tag{10*}$$
$$q(0) = -\int_0^T \left(C_b e^{-\int_0^\lambda r ds}\right) d\lambda. \tag{11*}$$

Let us to prove the following statements:

Statement 1: In the above-mentioned assumptions for F
$q(t) \geq 0$ for each $t \in [0, T]$.

Proof: We see, that $q(0) > 0$ is a positive number.

As
$$\frac{\partial}{\partial t}\left(\int_0^t F_b p e^{-\int_0^\lambda (r+F_b)ds} d\lambda\right) > 0,$$

Then
$$\int_0^t F_b p e^{-\int_0^\lambda (r+F_b)ds} d\lambda$$

is an increasing function of t.

Then
$$\int_0^T F_b p e^{-\int_0^\lambda (r+F_b)ds} d\lambda > \int_0^t F_b p e^{-\int_0^\lambda (r+F_b)ds} d\lambda$$

for each $t \in (0, T)$.

Therefore

$$q(t) = e^{\int_0^t (r+F_b)d\lambda} \left(\int_0^T F_b p e^{-\int_0^\lambda (r+F_b)ds} d\lambda - \int_0^t F_b p e^{-\int_0^\lambda (r+F_b)ds} d\lambda \right) > 0$$

for each $t \in (0, T)$. The Statement 1 is proven.

Statement 2: Let q is a decision of the conjugated system:

$$\dot{q} = rq - F_b(p - q) \ \ (\dot{q} = rq + C_b)$$
$$q(T) = 0, \tag{12}$$

and $p(t)$ is the market price of the product.

If $p(0) - q(0) > 0$ then $p(t) - q(t) > 0$ for each t.

Proof:

$$p(t) = p(0)e^{rt},$$

therefore p satisfies the condition

$$\dot{p} = rp.$$
$$\dot{q} = rq - F_b(p - q)$$

Then

$$\dot{p} - \dot{q} = (r - F_b)(p - q).$$

Therefore

$$(p - q)(t) = e^{\int_0^t (r-F_b)d\lambda}(p - q)(0) > 0 \ \forall \, t.$$

So, the Statement 2 is proven.

In this study we will formulated the following theorem 2 and will prove it using these two statements and the theorem Roll [31]:

Theorem 2: If $F(a,b): R^2 \to R$ is continuous and twice differentiable as $F_a > 0, F_{aa} < 0, F(0,b) = 0$ for each t; $p(t) = p(0)e^{rt}$; q is solution of the conjugated system (12) and

$$\lim_{a \to \infty} (p(O)e^{rt}F(a,b) - wa) \leq -\varepsilon < 0 \ \forall \, t$$

respectively

If $C(x,b): R^2 \to R$ is continuous and twice differentiable as $C_x > 0, C_{xx} < 0, C(0,b) = 0$ for each ; $p(t) = p(0)e^{rt}$; q is solution of the conjugated system (12) and

$$\lim_{x \to \infty} (p(O)e^{rt}x - C(x,b)) \leq -\varepsilon < 0 \ \forall \, t,$$

then $\widehat{H}(a,b,q)$ reaches its greatest value for $a \geq 0$ in a single point.

Proof: By Statement 1 follows, that $q(t) \geq 0 \ \forall t \in [0,T]$. Therefore, by

$$\lim_{a \to \infty} (pF(a,b) - wa) \ \leq -\varepsilon < 0 \ \ \forall \, t \in [0,T].$$

By $p(0) - q(0) > 0$ and Statement 2 follows that

$$p(t) - q(t) > 0 \, \forall \, t \in [0, T].$$

$F(a, b)$ is strictly concave of a and $p - q > 0$ for each t. Then $(p - q)F(a, b)$ is strictly concave of a.

Therefore $\widehat{H}(a, b, q)$ is strictly concave of a. $F(0, b) = 0$ for each b, then $\widehat{H}(a, b, q) = 0$ for each b and for each q. $\widehat{H}(a, b, q)$ is strictly concave of a, $\widehat{H}(0, b, q) = 0$ for each b and for each q and $\lim_{a \to \infty}((p - qF - wa) \leq -\varepsilon < 0$. It follows that, $\widehat{H}(a)$ crosses again the x −axis at the point a^{**} or does not cross the x− axis more than a point $a = 0$.

First case: $\widehat{H}(a, b, q)$ crosses the x− axis at the points $a = 0$ and $a = a^{**}$. So, $\widehat{H}(a, b, q) = 0 = \widehat{H}(a^{**}, b, q)$, $\widehat{H}(a, b, q)$ is continuous and differentiable function of a.

By the theorem of Roll follows, that exists $a^{**} \in (0, a^{**})$ such that:

$$\frac{\partial}{\partial a} \widehat{H}(a, b, q)\Big|_{a = a^*} = 0$$

Therefore

$$(p - q)F_a(a^*, b) = w.$$

Therefore $\widehat{H}(a, b, q)$ has local extremums. By strict concavity of $\widehat{H}(a, b, q)$ follows, that this local extremum is the only local extreme and is local maximum. $\widehat{H}(a^*, b, q)$ is the only local maximum of a for the continuous function $\widehat{H}(a, b, q)$ of $a \in [0, a^{**}]$, then it is the largest value of the function of a, $a \in [0, a^{**}]$.

$$\widehat{H}(a^*, b, q) > \widehat{H}(0, b, q) = 0 \, \forall \, b.$$

For $a > a^{**}$, the function of Hamilton becomes negative. Therefore $\widehat{H}(a^*, b, q)$ is the largest value of the function $\widehat{H}(a, b, q)$ of a, for $a \geq 0$.

Second case: $\widehat{H}(a, b, q)$ crosses the x− axis only in the point $a = 0$. As $\widehat{H}(0, b, q) = 0, \widehat{H}(0, b, q)$ is negative for sufficiently large values of $a, \widehat{H}(a, b, q)$ is strictly concave and continuous function of a, then $a = 0$ is the largest value of $\widehat{H}(a, b, q)$ of a, for each $a \geq 0$.

So, the theorem is proven.

Note: Let q is a solution of the conjugated system (12) and satisfies:

$$[(p-q)F(a^*,b^*)-wa^*]|_{t=T}=0$$
$$(p-q)F_a(a^*,b^*)-w=0$$

$$\begin{Bmatrix} [(p-q)x^*-C(x^*,b^*)]|_{t=T}=0 \\ p-q=C_x(x^*,b^*) \end{Bmatrix},$$

where $a^*(t)$ is an optimal control and $b^*(t)$ is its corresponding optimal trajectory. We find that $q=p-\frac{w}{F_a(a^*,b^*)}$.

After differentiation is obtained following:

$$\dot{q}=p\dot{q}=\dot{p}-\frac{w}{F_a}+\frac{\dot{w}F_{aa}}{F_a^2}\dot{a}+\frac{wF_{ab}}{F_a^2}\dot{b}-\frac{w}{F_a}+\frac{wF_{aa}}{F_a^2}\dot{a}+\frac{wF_{ab}}{F_a^2}\dot{b} \quad (13)$$
$$\dot{q}=rq-F_b(p-q)=r\frac{pF_a(a^*,b^*)-w}{F_a}-F_b(p-q). \quad (14)$$

As equate the right sides of (13) и (14), we get, that a^* is a solution of

$$\dot{a}=-\frac{F_a^2}{wF_{aa}}\dot{p}+\frac{F_a\dot{w}}{F_{aa}w}+\frac{F_{ab}}{F_{aa}}F+\frac{r(pF_a-w)F_a}{wF_{aa}}-F_b(p-q)\frac{F_a^2}{wF_{aa}},$$

$a(T): F_a\big(a(T),b(T)\big)=\frac{w}{p(T)}=\frac{w}{p_0e^{rt}}$, where $\dot{p}=rp, \dot{w}=rw$.

Analogously we get, that x^* is a solution of:

$$\dot{x}=-r\frac{p-C_x}{C_{xx}}-\frac{C_b}{C_{xx}}+\frac{C_{xb}}{C_{xx}}x+\frac{\dot{b}}{C_{xx}},$$
$$x(T): C_x\big(x(T),b(T)\big)=p(t)=p(0)e^{rt}.$$

4.2. Necessary and sufficient conditions for optimal management of mines

Let us formulate and prove the following theorem 3 based on the theorem 2.

Theorem 3: Let $b^*(t), q^*(t)$ are respectively the optimal path and the decision of the conjugated system of the task:

$$\min -\int_0^T[pF(a,b)-wa]e^{-rt}dt \quad (15)$$
$$b=-F(a,\dot{b}),\ \ b(0)=b_0$$

$F: R^2 \to R^+ \cup \{0\}$ is continuous and twice differentiable;

$$a\equiv a(t)\geq 0, b\equiv b(t)\in[0,b_0].$$

If

$$F(0,b)=0\ \forall b, F_a>0, F_{aa}<0, p(0)-q(0)>0, p=p_0e^{rt}$$

and

$$\lim_{a\to\infty}(p(O)e^{rt}F(a,b) - wa) \leq -\varepsilon < 0\ \forall t,$$

then the necessary and sufficient conditions $\breve{a}$ to be optimal control for (15) are:

(a) $\max_{a\geq 0}\widehat{H}(a,b^*,q^*) = H(\breve{a},b^*,q^*)$;

(b) $\widehat{H}(a,b^*,q^*)\big|_{t=T} = 0$.

Proof:

Necessary conditions:

Let $\breve{a}$ is an optimal control with a corresponding optimal trajectory b^*. By the principle of maximum Pontryagin follows that there is a decision q^* of the conjugated system problem (13) satisfying conditions (a) and (b) of the problem.

Sufficient conditions:

Let a^* is an optimal control of the problem (13) with a corresponding optimal trajectory b^*. Then a^* satisfies the conditions (a) and (b) of the problem. Let $\breve{a}$ is an control, satisfies the conditions (a) and (b) of the problem, but $\breve{a}$ is not optimal. By the theorem 2 follows that $\widehat{H}(a,b^*,q^*)$ reaches its greatest value of a for $a \geq 0$ into a single point i.e. $\breve{a} = a^*$ that is contrary to our assumption. Therefore $\breve{a}$ is an optimal control.

5. Existence of a solution of the problem for optimal control of mines of minerals

Consider the problem of the optimal control with a non-fixed end time moment T :

$$max\int_0^T (pF(a,b) - wa)e^{-rt}dt$$

$$\dot{b} = -F(a,b), b(0) = b_0 \qquad (16)$$

$$F: R^2 \to R^+ \cup \{0\},\ a \equiv a(t) \geq 0, w = w(t) \geq 0, p \equiv p(t).$$

Theorem 4: Let $F(a,b)$ is continuous and twice differentiable.

If $F(0,b) = 0$ for each b, $F_a > 0, F_{aa} < 0$, $p(0) - q(0) > 0$, $p = p_0e^{rt}, b \equiv b(t) \in [0,b_0]$ and $\lim_{a\to\infty}(p(0)\, e^{rt}F(a,b) - wa) \leq -\varepsilon < 0$ for each t, it exists only continuous $a^*(t,b)$, where in the function

$$V(a,b) = pF(a,b) - wa$$

reaches the its largest value for each t.

Proof: By $p(t) > 0$ for each t. It follows that $pF(a,b)$ is strictly concave of a.

Then $V(a,b) = pF(a,b) - wa$ is strictly concave of a. As $F(0,b) = 0$ for each b, $V(0,b) = 0$ for each b. $V(a,b)$ is strictly concave of a, $V(0,b) = 0$ for each b and

$$\lim_{a \to \propto} (p\,F(a,b) - wa) \leq -\varepsilon < 0.$$

Therefore $V(a,b)$ crosses again the x −axis at the point a^{**} or $V(a,b)$ does not cross again the x −axis more than the point $a = 0$.

First case: $V(a,b)$ crosses the x −axis for $a = 0$ and $a = a^{**}$.

a) As $V(0,b) = V(a^{**},b) = 0$ for each b, $V(a,b)$ is continuous and differentiable function of a ,by Roll's theorem follows that there is

$$a^* \in (0, a^{**}): \frac{\partial}{\partial x} V(a,b)\Big|_{a=a_*} = 0.$$

So

$$pF_a(a^*,b) = w. \qquad (17)$$

Therefore $V(a,b)$ has a local extreme. By strict concavity of $V(a,b)$ of a follows that the local extreme is an unique and is a local maximum.

$V(a^*,b)$ is the only local maximum of a for the continuous function $V(a,b)$ of a for $a^* \in (0,a^{**})$, then it is the greatest value of the function of a for $a^* \in [0,a^{**}]$, i.e. $V(a^*,b) > V(0,b) = 0$ for each b. For $a > a^{**}$, Hamilton's function becomes negative and therefore $V(a^*,b)$ is the largest value of the function $V(a,b)$ of a for each $a \geq 0$.

b) Let us consider the equation:

$$p(t)F_a(a^*,b) - w(t) = 0.$$

From a) follows that $p(t)F_a(a^*,b) - w(t) = 0$ for each b. As $p(t)F_{aa}(a^*,b) - w(t) \neq 0$ for each t and b, then by the implicit function theorem [31] follows that there is a single continuous $a^*(t,b)$, satisfying $p(t)F_a(a^*(t,b),b) - w(t) \equiv 0$ for each t and b. $a^*(t,b)$ is continuous. Therefore $a^*(t,b(t))$ will be also continuous for each continuous $b(t)$.

Second case: $V(a,b)$ crosses the x −axis only for $a = 0$. As $V(a,b) = 0$ for each b, $V(a,b)$ is negative for sufficiently large values of a, $V(a,b)$ is strictly concave and continuous of a, follows that

$a = 0$ is the greatest value of $V(a,b)$ of a for each $a \geq 0$.

$a^{**} \equiv 0$ is continuous function for each (t,b).

Consequence 1 of the theorem 4: If the conditions of the theorem 4 are satisfied, then the set of admissible controls is limited.

Proof: Theorem 4 shows that the optimal control belongs to the interval $[0, a^{**}]$ $(0 \le b \le b_0)$.

Let us prove, that $a^{**}(b)$ is limited for each $b \in [0, b_0]$. Let us assume that $a^{**}(b)$ tends to infinity for $0 \le b \le b_0$.

Then there is a limit of a sequence of points $b_k,\ k = 1,2, \ldots\ :\ 0 \le b_k \le b_0$ and $\lim_{k\to\infty} b_k = \bar{b} \in [0, b_0]$, for which $\lim_{k\to\infty} a^{**}(b_k) = \infty$.

It is known that a^{**} satisfies the equation $pF(a, \bar{b}) - wa = 0$.

Let us assume that $pF_a(a^{**}, \bar{b}) = w$. From theorem 4 follows that a^{*} is the only solution of this equation. Therefore $a^{*} = a^{**}$. Moreover, from the theorem 4 it follows that $V(a, \bar{b}) \ge 0$ for $a \in [0, a^{**}]$. Therefore $V(a, b) \equiv 0$ for $a \in [0, a^{**}]$. We received inconsistent with the strict concavity of $V(a, b)$. Whence it follows that $pF_a(a^{**}, \bar{b}) - w \ne 0$.

Then from the theorem of the implicit functions [31] we get that there is a single function $a^{**}(b)$, which is continuous in a neighborhood of the point $\bar{b}$:

$$pF(a^{**}(\bar{b}), \bar{b}) - wa^{**}(\bar{b}) = 0.$$

Therefore $\lim_{k\to\infty} a^{**}(b_k) = a^{**}(\bar{b}) = \infty$, which contradicts the continuity of a^{**} in a neighborhood of the $\bar{b}$. Therefore a^{**}(b) is limited for each $b \in [0, b_0]$.

For $a > a^{**}$, the function $V(a, b) = pF(a, b) - wa$ becomes negative (the profit of mine becomes negative) and the production should be terminated.

Theorem 5: (existing) Let $F(a, b)$ is continuous and twice differentiable, for which $F_a > 0, F_{aa} < 0, F(0, b) = 0$ for each b.

Let the following conditions are met:

1) $\lim_{a\to\infty} (pF(a, b) - wa) \le -\varepsilon < 0$;

2) For each b_0 there exist permissible control $a(t)$, for which exists $T_0(b_0): b(T_0) = 0$, where $\dot{b} = F(a(t), b), b(0) = b_0$;

3) $p(t) = p(0)e^{rt} > 0$.

Then there is optimal control $a^*(t)$ for the problem (16), which is the only solution of the equation $pF_a(a,b) = w$.

Note: The timing of the closure of the mine is $T \leq T_0$, as production is impossible without the presence of mineral reserves. In particular T_0 can be infinity.

Proof: As the conditions of Theorem 4 are present, it is applicable. From this theorem it follows that there is continuous $a^*\big(t, b(t)\big)$, which function $pF(a,b) - wa$ reaches its greatest value and $p(t)F_a\big(a^*, b(t)\big) = w(t)$.

Let us consider the function

$$u(t,a^*) = p(t)F\big(a^*, b(t)\big) - w(t)a^*(t) \text{ for each } b.$$

We will prove that $u(t,a^*) = \big[F\big(a^*(t), b(t)\big) - F_a\big(a^*, b(t)\big)a^*(t)\big]p(t)$ is limited for each t.

Let us assume that

$$\lim_{t\to\bar{t}} u(t,a^*) = \lim_{t\to\bar{t}} \big[F\big(a^*(t), b(t)\big) - F_a\big(a^*, b(t)\big)a^*(t)\big]p(t) = \infty.$$

Of continuity of $u(t,a^*)$ follows, that $\bar{t} = \infty$. Assuming $a^*(t)$ remains limited for $t \in [0,\infty)$, as $u(t,a^*)$ is continuous, $b(t) \in [0, b_0]$ will follow that $u(t,a^*)$ will remain constrained for infinitely large values of the time.

Therefore $\lim_{t\to\infty} a^*(t) = \infty$. So

$$\lim_{a^*(t)\to\infty} \big[F\big(a^*(t), b(t)\big) - F_a\big(a^*, b(t)\big)a^*(t)\big]p(t) = \infty.$$

This contradicts the condition 1) of the theorem. Then there is $M: u(t,a^*) \leq M$ for each t.

Therefore

$$\int_0^T u(t,a^*)e^{-rt}dt \leq \int_0^T Me^{-rt}dt = \frac{M}{r} - \frac{M}{r}e^{-rT} \leq \frac{M}{r}, \text{ for each } T < \infty.$$

In the case that T is infinity, then $\int_0^T u(t,a^*)e^{-rt}dt$ is convergent and therefore in this case there is an optimal control of the problem (16). So the theorem is proven.

6. Taxation

The contemporary countries have a tradition of regarding underground wealth as an eminent domain and a legitimate source of revenue. The simplest and most straightforward component is the bonus bid. “The bonus bid system [21] works best if the exploiting firm is more willing and able to bear risk. It is important, that there be genuine competition in the bidding. Jurisdictions prefer to bear some of the risk. The joint venture is a possible vehicle for doing this, but the country will levy taxes on the produced product, on profits or on rents”. As mentioned in [21], there are two main considerations in designing tax systems. The first is simply the yield and timing of the revenues, estimated over the predicted life of the mining operation. The second is attractive. A tax is said to be “better” if it is neutral. The most common tax is the gross royalty. The tax base is the market value of produced output and the tax is levied on an ad valorem basis. The royality will induce a “premature” end to the mining operation and the minerals will be left in the ground.

Let τ be the ad valorem gross royality. Then the present value of the mine can be calculated (in the dual) as

$$\int_0^T [(1-\tau)px - C(x,b)]e^{-rt}dt.$$

The price of the extracted material p is given. The total cost of the extraction is seen to depend on the remaining stock b as well as on the rate of the extraction x.

The current value of Hamiltonian is:

$$\widehat{H} = (1-\tau)px - qx - C(x,b).$$

The principle of maximum Pontryagin requires:

$$\widehat{H}\Big|_{t=T} = 0 \text{ (Hamiltonian must be zero at } T\text{)}$$

i.e. $[(1-\tau)px^* - qx^* - C(x^*,b^*)]|_{t=T} = 0$ and $(1-\tau)p - q = C_x(x^*,b^*)$.

But the marginal and average costs of the mining are rising as the stock of ore declines. q falls toward zero at T.

So $qb = 0$ at T. Then $q = 0, b > 0$ at T and some ore remains unmined. Therefore

$$(1-\tau)p = C_x(x^*,b^*) = \frac{C(x^*,b^*)}{x^*} \text{ at } T.$$

This means that the marginal and average costs will be lower at T, the higher τ is.

“The higher-grade cutoff problem [21] is averted for a producing mine by allowing the firm to write off its expenses from its pretax revenues. The firm will maximize both pretax and aftertax profits.” $(1-\tau)$ is a constant and it can be factored out of the integral and the problem of the mine is identical for all tax rates than one, and including zero. Therefore

$$V = (1-\tau)\int_0^T [px - C(x,b)]e^{-rt}dt.$$

The profit tax offers a theoretically satisfying solution to the problem of the mine. The unit price of the resource is q. If the profit of the mine is to be taxed at τ, the unit resource price is $(1-\tau)q$.

As mention in [21] the country collects a tax based on realized profits at τ and pays a subsidy $\sigma = -\tau$ when the mine has losses.

"The resource rent tax [21] is a radical departure from traditional methods of taxing resource rents." Let with E denote the expenses of the mine employing D units of effort to discover new reserves.

$$E = E(D), E'(D) > 0, E''(D) > 0.$$

Then the dynamic constraint is

$$\dot{b} = D - x.$$

For profit taxation, the mining firm is supposed to

$$\max \quad V = \int_0^T \{(1-\tau)[px - C(x,b)] - E(D)\}e^{-rt}dt.$$

For resource rent tax, the mine is supposed to

$$\max \quad V = \int_0^T (1-\tau)[px - C(x,b)] - E(D)]e^{-rt}dt.$$

The dynamic constraint is in both cases $\dot{b} = D - x$.

For profit taxation, Hamiltonian function is

$$\widehat{H} = (1-\tau)[px - C(x,b)] - E(D) + q(D-x).$$

For resource rent tax, this function is

$$\widehat{H} = (1-\tau)[px - C(x,b)E(D)] + q(D-x).$$

The mine has two control variables x, D and one state variable b. From the principle of the maximum of Pontryagin, for the profit taxation it follows that

$$(1-\tau)[p - C_x] - q = 0,$$
$$-E' + q = 0.$$

Therefore $(1-\tau)[p - C_x] = E'$ i.e. the price of the extracted product

$$p = C_x + \frac{E'}{1-\tau} = \text{marginal operating cost} + \text{marginal discovery cost}/(1-\tau)$$

When the tax rate τ decreases marginal operating cost, marginal discovery cost or both increases.

For the resource rent tax:

$(1-\tau)[p - C_x] - q = 0, (1-\tau)[-E'] + q = 0.$

Therefore $[p - C_x] = E^{'}$ i.e. the price of the extracted product
$p = C_x + E^{'} =$ marginal operating cost + marginal discovery cost.

7. Intuitionistic fuzzy model for an optimal control of a mine

In subsection 3.7 was defined a model for an optimal control of the mine of minerals. The modern market environment is distinguished by changeability and unstability. To handle with ambiguity Atanassov in [5] defined the concept of Intuitionistic fuzzy sets (IFS). We give some remarks on IFS (see [5,6]).

7.1. Basic definitions related with intuitionistic fuzzy sets

7.1.1. Intuitionistic fuzzy pairs (IFPs)

IFP is an object with the form $\langle a, b\rangle$, where $a, b \in [0,1]$ and $a + b \leq 1$, that is used as an evaluation of some object or process. Its components a and b are interpreted as degree of membership and non-membership. Let us have two IFPs $x = \langle a, b\rangle$ and $y = \langle c, d\rangle$.

The following relations are defined:

$$x < y \ iff \ a < c \text{ and } b > d, x \leq y \ iff \ a \leq c \text{ and } b \geq d,$$
$$x = y \ iff \ a = c \text{ and } b = d, x \geq y \ iff \ a \geq c \text{ and } b \leq d$$
$$\text{and } x > y \ iff \ a > c \text{ and } b < d.$$

Some of the operations with IFPs are:

$x\&y = \langle \min(a, c), \max(b, d)\rangle; x\vee y = \langle \max(a, c), \min(b, d)\rangle;$

$$x + y = \langle a + c - ac, bd\rangle; x.y = \langle ac, b + d - bd\rangle; x@y = \langle \frac{a+c}{2}, \frac{b+d}{2}\rangle.$$

In [6] are given many definitions of "negation". We recall following $\neg x = \langle b, a\rangle$.

7.1.2. Intuitionistic fuzzy set

Let a set E be fixed. An Intuitionistic fuzzy set (IFS) A in E is an object of the following form: $A = \{\langle x, \mu_A(x), \vartheta_A(x)\rangle | x \in E\}$, where functions $\mu_A: E \to [0,1]$ and $\vartheta_A: E \to [0,1]$ define respectively the degree of membership and the degree of non-membership of the element $x \epsilon E$ and for every $x \in E: 0 \leq \mu_A(x) + \vartheta_A(x) \leq 1$.

$$A \subset B \ iff \ (\forall x \in E)\big(\mu_A(x) \leq \mu_B(x) \& \vartheta_A(x) \geq \vartheta_B(x)\big);$$
$$A \supset B \ iff \ B \subset A; \bar{A} = \{\langle x, \vartheta_A(x), \mu_A(x)\rangle | x \in E\};$$
$$A = B \ iff \ (\forall x \in E)\big(\mu_A(x) = \mu_B(x) \& \vartheta_A(x) = \vartheta_B(x)\big);$$
$$A \cap B = \big\{\langle x, min\big(\mu_A(x), \mu_B(x)\big), max\big(\vartheta_A(x), \vartheta_B(x)\big)\rangle | x \in E\big\};$$
$$A \cup B = \big\{\langle x, max\big(\mu_A(x), \mu_B(x)\big), min\big(\vartheta_A(x), \vartheta_B(x)\big)\rangle | x \in E\big\};$$

7.1.3. Generalized Intuitionistic Fuzzy Number

Definition (6) [20]: An IFN $\tilde{A}^i$ is defined as follows:

(i) an intuitionistic fuzzy sybject of real line.

(ii) normal i.e. there is any $x_0 \epsilon R$ such that $\mu_{\tilde{A}^i}(x_0) = w, \nu_{\tilde{A}^i}(x_0) = \sigma$ for $0 < w + \sigma \leq 1$.

(iii) a convex set for the membership function $\mu_{\tilde{A}^i}(x)$ i.e. $\mu_{\tilde{A}^i}(\lambda x_1 + (1-\lambda)x_2) \geq min\left(\mu_{\tilde{A}^i}(x_1), \mu_{\tilde{A}^i}(x_2)\right) \forall x_1, x_2 \in R, \lambda \in [0, \omega]$

(iv) a concave set for the non-membership function $\nu_{\tilde{A}^i}(x)$ i.e. $\nu_{\tilde{A}^i}(\lambda x_1 + (1-\lambda)x_2) \geq min\left(\nu_{\tilde{A}^i}(x_1), \nu_{\tilde{A}^i}(x_2)\right) \forall x_1, x_2 \in R, \lambda \in [\sigma, 1]$

(v) $\mu_{\tilde{A}^i}$ and $\nu_{\tilde{A}^i}$ are continuous mappings from R to the closed interval $[0, \omega]$ and $[\sigma, 1]$ respectively and $x_0 \in R$, the relation $0 \leq \mu_{\tilde{A}^i} + \nu_{\tilde{A}^i} \leq 1$ holds.

7.2. Intuitionistic fuzzy integrals, posed by extended (fuzzy) integrals of Sugeno and Schocken

7.2.1. Basic definitions of generalized measure theory

Let X is nonempty set. Let C is nonempty set of subsets of X. $\mu: C \to [0, \infty]$ is negative and multiple extended real valued function defined on C. We recall some basic definitions from [29,30.]

Definition (6): Multiple function $\mu: C \to [0, \infty]$ is called summary measure on (X, C) if and only if $\mu(\emptyset) = 0$, if $\emptyset \in C$.

Definition (7): Multiple function $\mu: C \to [0, \infty]$ is called monotonous measure on (X, C) if and only if it satisfies following conditions:

(1) $\mu(\emptyset) = 0$, if $\emptyset \in C$;

(2) If $E \in C, F \in C$ and $E \subseteq F$ then $\mu(E) \leq \mu(F)$(monotony).

In [7] are defined intuitionistic fuzzy integrals based on fuzzy(extended) integrals of Sugeno and Schocken [28].

Here we recall Intuitionistic fuzzy integral of Schocken on finite sets following [7]:

7.2.2. Intuitionistic fuzzy integral of Schocken

Let $A^* = \langle x, \mu_A(x), \nu_A(x) \rangle | x \in E$, where $\mu_A: E \to [0,1]$ and $\nu_A: E \to [0,1]$ is IFS. $E = X = \{x_1, x_2, \ldots, x_n\}$.

Let $A^*_{interact} = \langle x, \mu_A(x), \nu_A(x)\rangle | x \in E$, where $\mu_A: E \to [0,1]$ and $\nu_A: E \to [0,1]$ is IFS with $E = P(X)$. $\mu_A(x)$ and $\nu_A(x)$ are summary measure (reset on empty set) on σ −algebra $P(X)$. $\mu_A(x)$ and $\nu_A(x)$ [7] are interpreted accordingly as no interaction rate and degree of interaction between the elements x_i, coming into x.

Definition (8): If $f: X \to [0,1]$, then

(IFC) $\int_X f dA^*_{interact} = \langle (C)\int_X f d\mu_A, (C)\int_X f d\nu_A\rangle$ is intuitionistic fuzzu integral of Schocken over final set X.

The meaning of above integral is that, when an evaluation of interaction A^*_i between x_i and a function of confidence gives fuzzy assessment of credibility of the experts of X.

7.2.3. Intuitionistic fuzzy integral of Schocken by function with fuzzy values on finite sets

Let $X = \{x_1, x_2, \dots, x_n\}$ and f is function, defined on X, with values in the set of fuzzy numbers. The function f can be expressed as $(m_1, m_2, \dots, m_n)$, where m_i is the function of belonging of fuzzy number $f(x_i), i = 1,2, \dots , n$.

Definition (9)[7]: For each $\alpha \in R$, the set α-level of fuzzy function $(f = m_1, m_2, \dots, m_n)$ will denote by F_α. It is fuzzy subset of X with function of belonging m_{F_α}, which is defined as follows:

$$m_{F_\alpha}(x_i) = \begin{cases} \dfrac{\int_\alpha^\infty m_i(t)dt}{\int_{-\infty}^{+\infty} m_i(t)dt}, & when \int_{-\infty}^{+\infty} m_i(t)dt \neq 0 \\ \max\limits_{t \geq \alpha} m_i(t), & in\ other\ cases \end{cases}$$

Fuzzy set F_α is n −dimensional vector $\left(m_{F_\alpha}(x_1), m_{F_\alpha}(x_1), \dots, m_{F_\alpha}(x_n)\right)$.

Let $A^* = \langle x, \mu_A(x), \nu_A(x)\rangle | x \in E$, where $\mu_A: E \to [0,1]$ and $\nu_A: E \to [0,1]$ is IFS with $E = P(X)$. $\mu_A(x), \nu_A(x)$ are summarize measures (reset on the empty set) on the σ-algebra $P(X)$. Let us $\tilde{P}(X)$ is the set of all fuzzy subsets of X.

Let $\tilde{A}^*_{interact} = \langle z, \tilde{\mu}_A(z), \tilde{\nu}_A(z)\rangle | z \in E$, where

$$\tilde{\mu}_A(z) = (C)\int_X \mu_z d\mu_A \text{ and } \tilde{\nu}_A(z) = (C)\int_X \nu_z d\nu_A,$$

is IFS with

$$E = \{z | z \in \tilde{P}(X) \text{ and } \mu_z \text{ is integrated of Schocken by } \mu_A \text{ and } \nu_A \text{ over } X\}.$$

In the integrals we use to define of $\tilde{\mu}_A(z), \tilde{\nu}_A(z)$, the function μ_z is function of belonging of fuzzy set z. Let the values of f are trapezoidal fuzzy numbers i.e.

$$f = (\langle a_{1l}, a_{1b}, a_{1c}, a_{1r}\rangle, \langle a_{2l}, a_{2b}, a_{2c}, a_{2r}\rangle, \dots, \langle a_{nl}, a_{nb}, a_{nc}, a_{1n}\rangle).$$

Then for each $\alpha \in R, \alpha-$level of f is fuzzy subset of X with function of belonging $m_{F_\alpha}(x_i)$ for $i = 1,2,\dots,n$. In this case as mentioned in [7] It is very difficult to get a manifest formula $\tilde{\mu}_A(F_\alpha)$ and $\tilde{\nu}_A(F_\alpha)$ by α. So in [7] are used numerical methods for calculus integrals and standard algorithm for computing the final integrals of **Schocken** to calculate approximately

$$(IFSC)\int_X f d\tilde{A}^*_{interact} = \langle \int_0^\infty \tilde{\mu}_A(F_\alpha)d\alpha, \int_0^\infty \tilde{\nu}_A(F_\alpha)d\alpha \rangle.$$

$\mu_A(x), \nu_A(x)$ – summirized measures обобщени мерки over $P(X)$, for which $\mu_A(L) + \nu_A(L) \leq 1$ for each $L \in P(X)$ and the values are negative numbers.

7.2.4. Intuitionistic fuzzy integral of Schocken by function with intuitionistic fuzzy values on finite sets

Let $X = \{x_1, x_2, \dots, x_n\}$ and f is function, defined on X, with values in the set of intuitionistic fuzzy numbers. The function f can be expressed as $(\langle \mu_1, \nu_1\rangle, \langle \mu_2, \nu_2\rangle, \dots, \langle \mu_n, \nu_n\rangle)$, where μ_i, ν_i are respectively functions of belonging and non-membership of intuitionistic fuzzy number $f(x_i), i = 1,2, \ ,n.$

Definition (10): For each $\alpha \in R$, the set α-level of intuitionistic fuzzy function $f = (\langle \mu_1, \nu_1\rangle, \langle \mu_2, \nu_2\rangle, \dots, \langle \mu_n, \nu_n\rangle)$ will denote by F_α. It is intuitionistic fuzzy subset of X with function of belonging and non-membership respectively μ_{F_α} and ν_{F_α}, which are defined as follows:

$$\mu_{F_\alpha}(x_i) = \begin{cases} \dfrac{\int_\alpha^\infty \mu_i(t)dt}{\int_{-\infty}^{+\infty} \mu_i(t) + 1 - \nu_i(t)dt}, & when \int_{-\infty}^{+\infty} \mu_i(t) + 1 - \nu_i(t)dt \neq 0 \\ \dfrac{1}{2} \max_{t \geq \alpha} \mu_i(t), & in\ other\ cases \end{cases}$$

$$\mu_{F_\alpha}(x_i) = \begin{cases} \dfrac{\int_\alpha^\infty 1 - \nu_i(t)dt}{\int_{-\infty}^{+\infty} \mu_i(t) + 1 - \nu_i(t)dt}, & when \int_{-\infty}^{+\infty} \mu_i(t) + 1 - \nu_i(t)dt \neq 0 \\ \dfrac{1}{2} \max_{t \geq \alpha} \big(1 - \nu_i(t)\big), & in\ other\ cases \end{cases}$$

Intuitionistic fuzzy set F_α can be expressed as follows $\big(\langle \mu_{F_\alpha}(x_1), \nu_{F_\alpha}(x_1)\rangle, \langle \mu_{F_\alpha}(x_2), \nu_{F_\alpha}(x_2)\rangle, \dots, \langle \mu_{F_\alpha}(x_n), \nu_{F_\alpha}(x_n)\rangle\big)$.

Let

$$A^* = \langle x, \mu_A(x), \nu_A(x)\rangle | x \in E,$$

where $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ is IFS with $E = X = \{x_1, x_2, \dots, x_n\}$.

Let

$$\tilde{A}^*_{interact} = \langle x, \mu_A(x), \nu_A(x)\rangle | x \in E,$$

where $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ is IFS with $E = P(X)$. $\mu_A(x), \nu_A(x)$ are summarize measures (reset on the empty set) on the σ-algebra $P(X)$.

Let us $P^*(X)$ is the set of all fuzzy subsets of X.

Let

$$\tilde{\tilde{A}}^*_{interact} = \langle z, \tilde{\tilde{\mu}}_A(z), \tilde{\tilde{\nu}}_A(z)\rangle | z \in E,$$

where $\tilde{\tilde{\mu}}_A(z) = \frac{1}{2}(C)\int_X \mu_z d\mu_A$ and $\tilde{\tilde{\nu}}_A(z) = \frac{1}{2}(C)\int_X \nu_z d\nu_A$, is IFS with $E = P^*(X)$.

In the integrals we use to define of $\tilde{\tilde{\mu}}_A(z), \tilde{\tilde{\nu}}_A(z)$, the functions μ_z and ν_z are respectively function of belonging and non-membership of intuitionistic fuzzy set z.

Let the values of f are trapezoidal intuitionistic fuzzy numbers i.e.

$f = (\langle a_{1l}, a_{1b}, a_{1c}, a_{1r}, a_{1l'}, a_{1b'}, a_{1c'} a_{1r'}\rangle, \langle a_{2l}, a_{2b}, a_{2c}, a_{2r}, a_{2l'}, a_{2b'}, a_{2c'}, a_{2r'}\rangle, \dots,$
$\langle a_{1l}, a_{1b}, a_{1c}, a_{1r}, a_{1l'}, a_{1b'}, a_{1c'} a_{1r'}\rangle)$.

Then for each $\alpha \in R, \alpha$ −level of f is intuitionistic fuzzy subset of X with function of belonging $\mu_{F_\alpha}(x_i)$ and function of non-membership $\nu_{F_\alpha}(x_i)$ for $i = 1,2, \dots, n$. In this case as mentioned in [7] It is very difficult to get a manifest formula $\tilde{\tilde{\mu}}_A(F_\alpha)$ and $\tilde{\tilde{\nu}}_A(F_\alpha)$ by α. So in [7] are used numerical methods for calculus integrals and standard algorithm for computing the final integrals of Schocken to calculate approximately

$$\int_X f d\tilde{\tilde{A}}^*_{interact} = \langle (C)\int_X f d\tilde{\tilde{\mu}}_A, (C)\int_X f d\tilde{\tilde{\nu}}_A\rangle = \langle \int_0^\infty \tilde{\tilde{\mu}}_A(F_\alpha)d\alpha, \int_0^\infty \tilde{\tilde{\nu}}_A(F_\alpha)d\alpha\rangle.$$

7.3. Intuitionistic fuzzy diferential equation

Melliani and Chadli [17,18] discussed differential and partial differential equations under intuitionistic fuzzy environment respectively. Abbasbandy and Viranloo [1] discussed numerical solution of fuzzy diferential equation by Runge-Kutta method with intuitionistic treatment. Lata and Kumar [13] have introduced time dependent intuitionistic fuzzy linear differential equation and have proposed a method to solve it. Mondal and Roy [19] has discussed strong and weak solution of intuitionistic fuzzy ordinary differential equation, in which they have solved first order homogeneous ordinary differential equation in intuitionistic fuzzy environment and discussed initial value as intuitionistic fuzzy number -- triangular intuitionistic fuzzy number. In [22] intuitionistic fuzzy Cauchy problem is solved numerically by Euler method under generalised differentiability concept. In [19] is described the generalized intuitionistic fuzzy laplace transform method for solving first order generalized intutionistic fuzzy differential equation. Condition in the presented problem is taken as Generalized Intuitionistic triangular fuzzy numbers (GITFNs).

Definition (10): The α-cut of an IFN $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in R\}$is defined as follows: $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in R, \mu_A(x) \geq \alpha \; and \; \nu_A(x) \leq 1-\alpha\}, \forall x \in [0,1]$. The α-cut of IFN A generates pair of intervals and is denoted by $[A]_\alpha = \{[A_L^+(\alpha), A_U^+(\alpha)], [A_L^-(\alpha), A_U^-(\alpha)]\}$.

Definition (11): Let $f: I \to \{IFN\, R^n\}$ for some interval I be an intuitionistic fuzzy function. The α-cut of f is given by

$$[f(t)]_\alpha = \left\{\left[\underline{f^+}(t,\alpha), \overline{f^+}(t,\alpha)\right], \left[\underline{f^-}(t,\alpha), \overline{f^-}(t,\alpha)\right]\right\}, \text{ where}$$

$$\underline{f^+}(t,\alpha) = Min\{f^+(t,\alpha) | t \in I, 0 \leq \alpha \leq 1\},$$

$$\overline{f^+}(t,\alpha) = Max\{f^+(t,\alpha) | t \in I, 0 \leq \alpha \leq 1\}$$

$$\underline{f^-}(t,\alpha) = Min\{f^-(t,\alpha) | t \in I, 0 \leq \alpha \leq 1\},$$

$$\overline{f^+}(t,\alpha) = Max\{f^-(t,\alpha) | t \in I, 0 \leq \alpha \leq 1\}$$

Definition (12): For arbitrary $u, v \in \{IFN\, R^n\}$, the

$$D(u,v) = \sup_{1 \leq \alpha \leq 1} d([u]^\alpha, [v]^\alpha)$$

is the distance between u and v [12], where d is the Hausdorf metric in $\{IFN\, R^n\}$ [12].

Definition (13)[8]: Let be $F:(a,b)\to\{IFN\,R^1\}$ and $x_0\epsilon(a,b)$. It is said that F is strongly generalized differentiable on x_0, if exist $F^{+'}(x_0), F^{-'}(x_0)\in\{FN\,R^1\}$, such that

(i) for all $h>0$ sufficiently small,

$$\exists F^+(x_0+h)-F^+(x_0), F^+(x_0)-F^+(x_0-h)$$

and the limits (in the metric D)

$$\lim_{h\searrow 0}\frac{F^+(x_0+h)-F^+(x_0)}{h}=\lim_{h\searrow 0}\frac{F^+(x_0)-F^+(x_0-h)}{h}=F^{+'}(x_0)$$

OR

(ii) for all $h>0$ sufficiently small,

$$\exists F^+(x_0)-F^+(x_0+h), F^+(x_0-h)-F^+(x_0)$$

and the limits $\lim_{h\searrow 0}\frac{F^+(x_0-h)-F^+(x_0)}{h}=\lim_{h\searrow 0}\frac{F^+(x_0)-F^+(x_0+h)}{-h}=F^{+'}(x_0)$

OR

(iii) for all $h>0$ sufficiently small,

$$\exists F^+(x_0+h)-F^+(x_0), F^+(x_0-h)-F^+(x_0)$$

and the limits

$$\lim_{h\searrow 0}\frac{F^+(x_0+h)-F^+(x_0)}{h}=\lim_{h\searrow 0}\frac{F^+(x_0-h)-F^+(x_0)}{-h}=F^{+'}(x_0)$$

OR

(iv) for all $h>0$ sufficiently small,

$$\exists F^+(x_0)-F^+(x_0+h), F^+(x_0)-F^+(x_0-h)$$

and the limits

$$\lim_{h\searrow 0}\frac{F^+(x_0)-F^+(x_0+h)}{-h}=\lim_{h\searrow 0}\frac{F^+(x_0)-F^+(x_0-h)}{-h}=F^{+'}(x_0)$$

Where h and $-h$ at deniminators mean $\frac{1}{h}\odot$ and $-\frac{1}{h}\odot$ respectively. Results similar from *(i)* to *(iv)* can be defined to the function $F^{-'}(x_0)\in FN\,R^1$.

A first order intuitionistic fuzzy diferential equation[16] is a diferential equation of the form:

$$y'(t)=f\big(t,y(t)\big),$$
$$y(t_0)=y_0$$
$$t\epsilon I=[a,b],$$

where y is an intuitionistic fuzzy function of the crisp variable t; $f(t,u)$ is an intuitionistic fuzzy function of the crisp variable t and $y'(t)$ is the intuitionistic fuzzy derivative.

If an initial value $y(t_0)=y_0$\{intuitionistic fuzzy number\}, then we get an intuitionistic fuzzy Cauchy problem of first order

$$y' = f(t,y), y(t_0) = y_0. \quad (18)$$

Each intuitionistic is a conjecture two fuzzy numbers [10], equation (18) is replaced by:

$$y'(t) = \left\{\left[\underline{y'^+}(t), \overline{y'^+}(t)\right], \left[\underline{y'^-}(t), \overline{y'^-}(t)\right]\right\}, \text{where}$$

$$\underline{y'^+}(t) = \underline{f^+}(t,y^+) = min\left\{f^+(t,u) | u \in \left[\underline{y^+}, \overline{y^+}\right]\right\} = F\left(t, \underline{y^+}, \overline{y^+}\right),$$
$$\underline{y^+}(t_0) = \underline{y_0^+} \quad (19)$$

$$\overline{y'^+}(t) = \overline{f^+}(t,y^+) = max\left\{f^+(t,u) | u \in \left[\underline{y^+}, \overline{y^+}\right]\right\} = G\left(t, \underline{y^+}, \overline{y^+}\right),$$
$$\overline{y^+}(t_0) = \overline{y_0^+} \quad (20)$$

$$\underline{y'^-}(t) = \underline{f^-}(t,y^-) = min\left\{f^-(t,u) | u \in \left[\underline{y^-}, \overline{y'^-}\right]\right\} = H\left(t, \underline{y^-}, \overline{y^-}\right),$$
$$\underline{y^-}(t_0) = \underline{y_0^-} \quad (21)$$

$$\overline{y'^-}(t) = \overline{f^-}(t,y^-) = max\left\{f^-(t,u) | u \in \left[\underline{y^-}, \overline{y^-}\right]\right\} = I\left(t, \underline{y^-}, \overline{y^-}\right),$$
$$\overline{y^-}(t_0) = \overline{y_0^-} \quad (22)$$

The system of equations in (19) and (20) have unique solution

$$\left[\underline{y^+}, \overline{y^+}\right] \in B = \overline{C}[0,1] \times \overline{C}[0,1]$$

and the system of equations (21) and (22) have unique solution

$$\left[\underline{y^-}, \overline{y^-}\right] \in B = \overline{C}[0,1] \times \overline{C}[0,1].$$

So the system given from (19) to (22) possesses unique solution

$$y(t) = \left\{\left[\underline{y^+}(t), \overline{y^+}(t)\right], \left[\underline{y^-}(t), \overline{y^-}(t)\right]\right\} \in B \times B,$$

which is an intuitionistic fuzzy function; for each

$$t, y(t,r) = \left\{\left[\underline{y^+}(t,r), \overline{y^+}(t,r)\right], \left[\underline{y^-}(t,r), \overline{y^-}(t,r)\right]\right\}, r \in [0,1]$$

is an intuitionistic fuzzy number.

The parametric form of the system of equations (19)-(22) is given by:

$$\underline{y'^+}(t,r) = F\left(t, \underline{y^+}(t,r), \overline{y^+}(t,r)\right); \underline{y^+}(t_0,r) = \underline{y_0^+}(r) \quad (23)$$

$$\overline{y'^+}(t,r) = G\left(t, \underline{y^+}(t,r), \overline{y^+}(t,r)\right); \overline{y^+}(t_0,r) = \overline{y_0}^+(r) \quad (24)$$

$$\underline{y'^{-}}(t,r) = \mathrm{X}\left(t, \underline{y^{-}}(t,r), \overline{y^{-}}(t,r)\right); \underline{y^{-}}(t_0,r) = \underline{y_0^{-}}(r) \quad (25)$$

$$\overline{y'^{-}}(t,r) = \text{И}\left(t, \underline{y^{-}}(t,r), \overline{y^{-}}(t,r)\right); \overline{y^{-}}(t_0,r) = \overline{y_0}^{-}(r) \quad (26)$$

for $r \in [0,1]$.

A solution to the system of equations (23)-(26) solves equations(19) to(22). As mentioned in [22], in some cases, the system (23)-(26) can be solved analytically. But in most cases, however, analytical solutions to equations (23)-(26) do not be found and a numerical approach must be considered. For every r, each equation from (23)-(26) represents an ordinary Cauchy problem for which any numerical procedure can be applied. In [22] also is proposed Euler method and a complete error analysis guarantees the method's convergence to the unique solution to equation (18).

To solve the intuitionistic fuzzy system of ordinary differential system in $[t_0; t_1]; [t_1; t_2]; \dots [t_k; t_{k+1}]; \dots$ each interval $[t_k; t_{k+1}]$ is replaced by a set of $N_k + 1$regularly spaced points.

The grid points on $[t_k; t_{k+1}]$ will be

$$t_{k,n} = t_k + nh_k,$$

where

$$h_k = \frac{t_{(k+1)} - t_k}{t_1}$$

and $0 \le n \le N_k$.

The defined algorithm approximate the solution of the fuzzy initial value problem given by the system of equations (23)-(24)(or (25)-(26)) in the cases of *{(i),(ii)}*-diferentiability.

7.4. Generalized Intutionistic Fuzzy Laplace transform (GIFLT)

Laplace transform is a very useful apparatus to solve differential equation. The Laplace transforms give the solution of a differential equations satisfying the initial condition directly without use the general solution of the differential equation. Fuzzy Laplace Transform was first introduced by Allahviranloo and Ahmadi [2]. In [20] are performed following definitions and theorems.

Definition (14): Non negative GTIFN (Generalized Triangular Intuitionistic Fuzzy Number) is

$$\tilde{A}^{i}_{GTIFN} = \left((a_1, a_2, a_{3,}\omega), (a'_1, a_2, a'_3, \sigma)\right) \; iff \; a'_1 \ge 0.$$

Theorem 6: Let $f(x)$ be a generalized intuitionistic fuzzy number valued function on $[a,\infty]$ and it represented by $(f_1(x,\alpha;\omega), f_2(x,\alpha;\omega), g_1(x,\beta;\sigma), g_2(x,\beta;\sigma))$, where $\alpha \in [0,\omega], \beta \in [\sigma,1], 0 \leq \omega, \sigma \leq 1$. Assume $f_1(x,\alpha;\omega), f_2(x,\alpha;\omega), g_1(x,\beta;\sigma)$ and $g_2(x,\beta;\sigma)$ are Riemann-integrable on $[a,b]$ for every $b \geq a$, and assume, there are four positive function $M_1(\alpha), M_2(\alpha), N_1(\beta)$ and $N_2(\beta)$, such that,

$$\int_a^b |f_1(x,\alpha;\omega)| dx \leq M_1(\alpha), \int_a^b |f_2(x,\alpha;\omega)| dx \leq M_2(\alpha),$$

$$\int_a^b |g_1(x,\beta;\omega)| dx \leq N_1(\beta)$$

and

$$\int_a^b |g_2(x,\beta;\omega)| dx \leq N_2(\beta)$$

for every $b \geq a$. Then $f(x)$ is an intuitionistic fuzzy Riemann-integrable on $[a,\infty)$ and the intuitionistic fuzzy Riemann-integral is an intuitionistic fuzzy number. So,

$$\int_a^\infty f(x)dx = \left(\int_a^\infty f_1(x,\alpha,\omega)\, dx, \int_a^\infty f_2(x,\alpha,\omega)\, dx, \int_a^\infty g_1(x,\beta,\sigma)\, dx, \int_a^\infty g_2(x,\beta,\sigma)\, dx \right).$$

Theorem 7: Let $f(x)$ be a continuous intuitionistic fuzzy valued function. Suppose that $f(x) \odot e^{-px}$ is improper fuzzy Riemann-integrable on $[0,\infty)$, then $\int_0^\infty f(x) \odot e^{-px} dx$ is called intuitionistic fuzzy Laplace transforms and is denoted by: $L[f(x)] = \int_0^\infty f(x) \odot e^{-px} dx (p > 0$ and integer). Then

$$\int_a^\infty f(x) \odot e^{-px}\, dx$$

$$= \left(\int_a^\infty f_1(x,\alpha,\omega) \odot e^{-px}\, dx, \int_a^\infty f_2(x,\alpha,\omega) \odot e^{-px}\, dx, \int_a^\infty g_1(x,\beta,\sigma) \odot e^{-px}\, dx, \int_a^\infty g_2(x,\beta,\sigma) \odot e^{-px}\, dx \right).$$

Theorem 8: Let $f(x)$ be a continuous intuitionistic fuzzy valued function and $L[f(x)] = F(p)$, then $L[e^{ax} \odot f(x)] = F(p-a)$, where e^{ax} is real valued function and $p - a > 0$.

Theorem 9: Let $f: R \to E$ be a function and denote $f(x) = (f_1(x,\alpha;\omega), f_2(x,\alpha;\omega), g_1(x,\beta;\sigma), g_2(x,\beta;\sigma))$ for each $\alpha \in [0,\omega], \beta \in [\sigma,1], 0 \leq \omega, \sigma \leq 1$ then

(1) If f is $(i) - gH$ differentiable then $f_1(x,\alpha;\omega), f_2(x,\alpha;\omega), g_1(x,\beta;\sigma)$ and $g_2(x,\beta;\sigma)$ are differentiable functions and $f'(x) = \left(f_1{}'(x,\alpha;\omega), f_2{}'(x,\alpha;\omega), g_1{}'(x,\beta;\sigma), g_2{}'(x,\beta;\sigma)\right)$

(2) If f is $(ii) - gH$ differentiable then $f_1(x,\alpha;\omega), f_2(x,\alpha;\omega), g_1(x,\beta;\sigma)$ and $g_2(x,\beta;\sigma)$ are differentiable functions and $f'(x) = \left(f_2{}'(x,\alpha;\omega), f_1{}'(x,\alpha;\omega), g_2{}'(x,\beta;\sigma), g_1{}'(x,\beta;\sigma)\right)$.

Theorem 10: Let $f'(x)$ be an integrable fuzzy valued function, and $f(x)$ is the primitive of $f'(x)$ on $[0,\infty)$. Then

$$L[f'(x)] = p \odot Lf(x) -^h f(0),$$

when f is $(i) - gH$ differentiable and $L[f'(x)] = (-f(0)) -^h (-p \odot L[f(x)])$, when f is $(ii) - gH$ differentiable

7.5. Intuitionistic fuzzy model for the optimal control of the mines

In [20] is given procedure for solution the first order linear generalized intutionistic fuzzy differential equation by Generalized intutionistic fuzzy laplace transform method. Then this procedure is applied in two different imprecise problems.

In subsection (3.7) were defined folowing optimal problems:

$$min - \int_0^T (pF(a,b) - wa)e^{-rt} dt$$

$$\dot{b} = -F(a,b), \quad b(t) \in [0, b_0], a(t) \geq 0 \, for \, 0 \leq t \leq t_2.$$

OR

$$\min - \int_0^T (px - C(x,b))e^{-rt} dt$$

$$\dot{b} = -x; \; b(0) = b_0; \; b(t) \in [0, b_0], \; x(t) \geq 0 \, for \, 0 \leq t \leq t_2.$$

From the principle of maximum of Pontryagin we obtain that [32,33,35], if $a^*(t)(x^*(t))$ is optimal control $(0 \leq t \leq T)$ with corresponding decision $\hat{b}^*(t) = (b^{0*}(t), b^*(t))$, it exists nontrivial solution q of

$$\dot{q} = rq - F_b(p - q), \ \ (\dot{q} = rq + C_b),$$
$$q(T) = 0, \quad (q(T) = 0).$$

A first order intuitionistic fuzzy diferential equation[16] is a diferential equation of the form:

$$y'(t) = f\big(t, y(t)\big),$$
$$y(t_0) = y_0$$
$$t\epsilon I = [a, b],$$

where if q is an intuitionistic fuzzy function of the crisp variable t; $rq - F_b(p - q)$, $(\dot{q} = rq + C_b)$ is an intuitionistic fuzzy function of the crisp variable t and $q'(t)$ is the intuitionistic fuzzy derivative. If an initial value $q(t_0) = q_0$\{intuitionistic fuzzy number\}, then we get an intuitionistic fuzzy Cauchy problem of first order.

The solution of the problem of Cauchy for intuitionistic fuzzy function q can be implemented on the presented methods as the requirement for initial value for the method of GIFLT is to be Generalized Intuitionistic triangular fuzzy numbers

$$\tilde{A}^i_{GTIFN} = \Big(\big(a_1, a_2, a_{3,}\omega\big), (a'_1, a_2, a'_3, \sigma)\Big).$$

An interesting case of the optimal problem is when the Riemann integral in it is an Intuitionistic fuzzy integral of Schoken. The marked ideas of intuitionistic modells will be the subject of further research.

8. Conclusion

As mentioned in [9] “Minining models continue to be developed, refined and implemented, and we predict that the use of operations research will become increasingly significant as mathematical modeling and technology, while the demand for natural resources increases and resources themselves diminish.”

In this study was proved the existence of a solution of the problem for an optimal control of a mine of resourses. In it were pointed algorithms for finding a solution of intuitionistic fuzzy models of a problem of mine’ s optimal control. These models are more effective in the changing market environment.

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